These values satisfy condition (2).

Substituting in (1) we have

$$a^2-b^2=1\ldots.(3).$$

If e is the eccentricity of the ellipse $b^2=a^2(1-e^2)$, substituting in (3) and reducing, we have $ae=\pm1$, *i. e.* the foci of the ellipse are at the points ±1.

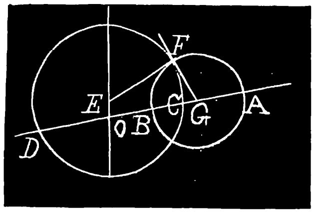
Also solved by *G. B. M. ZERR*, and the *PROPOSER*.

140. **Proposed by J. OWEN MAHONEY, B. E., M. Sc., Professor of Mathematics, Central High School, Dallas, Tex.**

Having given two points on a range and a point that bisects the distance between two other points that form an harmonic ratio with the given points, give, if possible, a geometrical construction for locating the other two points.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let A, B be the two given points on the range, O the point bisecting the distance between the other two points. Through O draw OE perpendicular to ABO, and on AB as diameter describe a circle AFB. Draw EF tangent to AFB, and with E as center and a radius equal to EF describe a circle cutting AO in C and D. (F is the point of tangency of EF, and EF must be greater than EO). Then C, D are the two points required.



For $GF^2=AG^2=GC.GD$.

$\therefore$ $AG:GC=GD:AG$.

But $AG+GC:AG-GC=GD+AG:GD-AG$.

$\therefore$ $AC:CB=AD:BD$. Q. E. D.

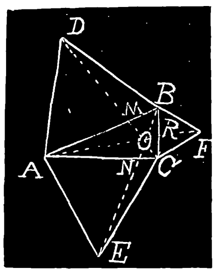
141. **Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.**

The equilateral triangle described on the hypotenuse of a right triangle is equivalent to the sum of the equilateral triangles described on the other two sides.

Prove without the aid of the famous Pythagorean proposition.

Solution by J. SCHEFFER, A. M., Hagerstown, Md., and NELSON L. RORAY, Bridgeton, N. J.

Let ABC represent the right triangle, and D, E, F the vertices of the equilateral triangles constructed on the three sides.



It is seen at once that $\triangle ACF=\triangle BCE=\frac{1}{2}\triangle ABC$.

$\therefore$ $\triangle ACF+\triangle BCE=\triangle ABC$.

$\triangle BDC=\triangle ABF$, $\triangle ADC=\triangle AEB$.

$\therefore$ $\triangle ABD+\triangle ABC=\triangle ABF+\triangle AEB$.

$\triangle ABC=\triangle ACF+\triangle BCE$.

$\therefore$ $\triangle ABD+2\triangle ABC=(\triangle ABF+\triangle ACF)+(\triangle AEB+\triangle BCE)$.

$\therefore$ $\triangle ABD+2\triangle ABC=\triangle BCF+\triangle ABC+\triangle ACE+\triangle ABC$.

$\therefore\ \triangle ABD = \triangle BCF + \triangle ACE.$ Q. E. D.

That the three lines AF, BE, and CD intersect in one point may be proved as follows:

Since EOB cuts the sides of $\triangle ACM$ in the points N, O, B, $CN\times OM \times AB = AN\times CO\times BM\ldots.(1)$.

Since AOF cuts the sides of $\triangle BCM$ in O, P, F, $CO\times BP\times AM = OM\times CP \times AP\ldots.(2)$.

Multiplying (1) by (2) and canceling, $CN\times BP\times AM = AN\times BM\times CP$.

Also demonstrated by *G. B. M. ZERR*, and the *PROPOSER*.

142. Proposed by WILLIAM HOOVER. A.M., Ph.D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Show that an infinite number of triangles can be inscribed in $x^2/a^2+y^2/b^2-1=0$ whose sides touch $a^2x^2+b^2y^2=\dfrac{a^4b^4}{(a^2+b^2)^2}$.

I. Solution by the PROPOSER.

The curves are $b^2x^2+a^2y^2-a^2b^2=0\ldots.(1)$,

$$\text{and } a^2(a^2+b^2)^2x^2+b^2(a^2+b^2)^2y^2-a^4b^4=0\ldots.(2).$$

The invariants of (1) are $\triangle = -a^4b^4$,

$$\theta' = -a^2b^2(a^4+a^2b^2+b^4)^2$$

and of (2), $\triangle = -a^6b^6(a^2+b^2)^4$,

$$\theta' = -2a^4b^4(a^2+b^2)^2(a^4+a^2b^2+b^4).$$

By the usual theory, the conditions of the problem are fulfilled if $\theta^2=4\triangle\theta'\ldots.(3)$, which is easily seen to be the case here.

II. Solution by GEORGE A. OSBORNE, Professor of Mathematics, Massachusetts Institute of Technology, Boston, Mass.

The problem is a special case of the following:

An infinite number of triangles can be inscribed in the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ whose sides touch $\dfrac{x^2}{a'^2}+\dfrac{y^2}{b'^2}=1$, provided $\dfrac{a'}{a}+\dfrac{b'}{b}=1$.

The problem is considered in Salmon's *Conic Sections*, Art. 376, page 342, for the general form of the conic.

$$S = ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.$$

$$S' = a'x^2+b'y^2+c'z^2+2f'yz+5g'zx+2h'xy=0.$$

The condition that an infinite number of triangles may be inscribed in S and circumscribed about S', is $\Theta'^2=4\Delta'\Theta\ldots.(1)$.

Δ, Δ', are the discriminants of S, S'.